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Quaternionic factorization of the Schrödinger operator and its applications to some first-order systems of mathematical physics

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Abstract

We show that an ample class of physically meaningful partial differential systems of first order such as the Dirac equation with different one-component potentials, static Maxwell's system and the system describing the force-free magnetic fields are equivalent to a single quaternionic equation which in its turn reduces in general to a Schrödinger equation with quaternionic potential, and in some situations this last can be diagonalized. The rich variety of methods developed for different problems corresponding to the Schrödinger equation can be applied to the systems considered in the present work.

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1. Introduction

We consider the following first-order systems of mathematical physics.

1. The Dirac equation with scalar potential.
2. The Dirac equation with electric potential.
3. The Dirac equation with pseudoscalar potential.
4. The system describing non-linear force-free magnetic fields or Beltrami fields with nonconstant proportionality factor.
5. The Maxwell equations for slowly changing media.
6. The static Maxwell system.

We show that all this variety of first-order systems reduces to the equation

$$Df + f \cdot \vec{\alpha} = 0 \tag{1}$$

where D is the Moisil–Theodoresco operator (introduced by Hamilton) acting on biquaternion valued functions f according to the rule $Df = \sum_{k=1}^{3} e_k \partial_k f$, $\partial_k = \frac{\partial}{\partial x_k}$, e_k are standard quaternionic imaginary units; the function f of real variables x_1, x_2, x_3 has the form $f = \sum_{k=0}^{3} f_k e_k$, $f_k \in C, k = 0, 1, 2, 3$ and $\vec{\alpha}$ is a purely vectorial biquaternion valued function. We reduce the solution of equation (1) to the solution of a Schrödinger equation with biquaternionic potential. In some important situations the biquaternionic potential can be diagonalized and converted into scalar potentials.

2. Notation

We will consider the algebra $\mathbb{H}(\mathbb{C})$ of complex quaternions or biquaternions which have the form $q = \sum_{k=0}^{3} q_k e_k$ where $\{q_k\} \subset \mathbb{C}$, e_0 is the unit and $\{e_k \mid k = 1, 2, 3\}$ are the quaternionic imaginary units, that is the standard basis elements possessing the following properties:

$$e_0^2 = e_0 = -e_k^2 \qquad e_0 e_k = e_k e_0 = e_k \qquad k = 1, 2, 3$$
$$e_1 e_2 = -e_2 e_1 = e_3 \qquad e_2 e_3 = -e_3 e_2 = e_1 \qquad e_3 e_1 = -e_1 e_3 = e_2.$$

We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with $e_k, k = 0, 1, 2, 3$.

The vectorial representation of a complex quaternion will be used. Namely, each complex quaternion q is a sum of a scalar q_0 and of a vector $\vec{q}$:

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q) = q_0 + \vec{q}$$

where $\vec{q} = \sum_{k=1}^{3} q_k e_k$. The purely vectorial complex quaternions ($\mathrm{Sc}(q) = 0$) are identified with vectors from $\mathbb{C}^3$. The following conjugation operations will be needed. The quaternionic conjugation is defined as follows $\bar{q} = q_0 - \vec{q}$, the complex conjugation: $q^* = \mathrm{Re}\, q - \mathrm{i}\, \mathrm{Im}\, q$ and the following involutive operation $q^{(k)} = e_k q \overline{e_k}$ which changes signs of two components, for example, $q^{(1)} = -e_1 q e_1 = q_0 e_0 + q_1 e_1 - q_2 e_2 - q_3 e_3$.

By M^p we denote the operator of multiplication by a complex quaternion p from the right-hand side: $M^p q = q \cdot p$.

We will intensively use the fact that the algebra of complex quaternions contains a subset of zero divisors $\mathfrak{S}$ which are characterized by the equality $q_0^2 = \vec{q}^2$, where $\vec{q}^2 = -\langle \vec{q}, \vec{q} \rangle$, or equivalently $q^2 = 2q_0 q$. Hence if $q \in \mathfrak{S}$ and $q_0 = 1/2$ then q is an idempotent. Using this fact we introduce the following multiplication operators $P_k^{\pm} = \frac{1}{2} M^{(1 \pm \mathrm{i} e_k)}$, $k = 1, 2, 3$. It is easy to see that for each k the operators P_k^+ and P_k^- represent a pair of mutually complementary, orthogonal projection operators on the set of $\mathbb{H}(\mathbb{C})$-valued functions. More information on the structure of the algebra of complex quaternions can be found for example in [17] or [19].

Let f be a complex quaternion valued differentiable function of $\mathbf{x} = (x_1, x_2, x_3)$. Denote

$$Df = \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k} f.$$

This expression can be rewritten in vector form as follows:

$$Df = -\mathrm{div}\, \vec{f} + \mathrm{grad}\, f_0 + \mathrm{rot}\, \vec{f}.$$

That is, $\mathrm{Sc}(Df) = -\mathrm{div} \vec{f}$ and $\mathrm{Vec}(Df) = \mathrm{grad} f_0 + \mathrm{rot} \vec{f}$. Let us note that $D^2 = -\Delta$. The operator $D + M^{\vec{\alpha}}$ we will denote also by $D_{\vec{\alpha}}$.

Let us introduce an auxiliary notation $\tilde{f} := f(x_1, x_2, -x_3)$. The domain $\tilde{\Omega}$ is assumed to be obtained from the domain $\Omega \subset \mathbb{R}^3$ by the reflection $x_3 \to -x_3$.

3. First-order systems reducing to equation (1)

3.1. The Dirac equation with scalar potential

Let $\mathcal{D}$ denote the classic Dirac operator for a free particle with a specified energy $\omega \in \mathbb{R}$

$$\mathcal{D} = \mathrm{i}\omega\gamma_0 + \sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}m.$$

Here γ_j, $j = 0, 1, 2, 3$ are usual γ-matrices (see, e.g., [5]) and $m \in \mathbb{R}$.

The Dirac operator with scalar potential has the form (see, e.g., [23])

$$\mathcal{D}^{\mathrm{sc}} = \mathcal{D} + \mathrm{i}\varphi_{\mathrm{sc}} I$$

where φ_{sc} is a scalar real-valued function of $\mathbf{x}$ and I denotes the identity operator.

In [13] (see also [17, 19]) the following transformation was introduced. A function $\Phi : \Omega \subset \mathbb{R}^3 \to \mathbb{C}^4$ is transformed into a function $F : \tilde{\Omega} \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule

$$F = \mathcal{A}[\Phi] := \tfrac{1}{2}(-(\tilde{\Phi}_1 - \tilde{\Phi}_2)e_0 + \mathrm{i}(\tilde{\Phi}_0 - \tilde{\Phi}_3)e_1 - (\tilde{\Phi}_0 + \tilde{\Phi}_3)e_2 + \mathrm{i}(\tilde{\Phi}_1 + \tilde{\Phi}_2)e_3).$$

The inverse transformation $\mathcal{A}^{-1}$ is defined as follows:

$$\Phi = \mathcal{A}^{-1}[F] = (-\mathrm{i}\tilde{F}_1 - \tilde{F}_2, -\tilde{F}_0 - \mathrm{i}\tilde{F}_3, \tilde{F}_0 - \mathrm{i}\tilde{F}_3, \mathrm{i}\tilde{F}_1 - \tilde{F}_2).$$

Let us present the introduced transformations in a more explicit matrix form which relates the components of a $\mathbb{C}^4$-valued function Φ with the components of an $\mathbb{H}(\mathbb{C})$-valued function F:

$$F = \mathcal{A}[\Phi] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ -1 & 0 & 0 & -1 \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\Phi}_0 \\ \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \tilde{\Phi}_3 \end{pmatrix}$$

and

$$\Phi = \mathcal{A}^{-1}[F] = \begin{pmatrix} 0 & -\mathrm{i} & -1 & 0 \\ -1 & 0 & 0 & -\mathrm{i} \\ 1 & 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{pmatrix}.$$

The following equality is valid

$$D_{\vec{\alpha}_{\mathrm{sc}}} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{D}^{\mathrm{sc}}\mathcal{A}^{-1} \tag{2}$$

where $D_{\vec{\alpha}_{\mathrm{sc}}} = D + M^{\vec{\alpha}_{\mathrm{sc}}}$ and $\vec{\alpha}_{\mathrm{sc}} = -(\mathrm{i}\omega e_1 + (m + \tilde{\varphi}_{\mathrm{sc}})e_2)$. Thus the function Φ is a solution of the Dirac equation with scalar potential

$$\mathcal{D}^{\mathrm{sc}}\Phi = 0 \qquad \text{in} \quad \Omega$$

if and only if the function $F = \mathcal{A}\Phi$ is a solution of the equation

$$D_{\vec{\alpha}_{\mathrm{sc}}} F = 0 \qquad \text{in} \quad \tilde{\Omega}.$$

3.2. The Dirac equation with electric potential

The Dirac equation with electric potential has the form

$$\mathcal{D}^{\mathrm{el}} = \mathcal{D} + \mathrm{i}\varphi_{\mathrm{el}}\gamma_0$$

where φ_{el} is a real-valued function.

We have an equality similar to (2):

$$D_{\vec{\alpha}_{\mathrm{el}}} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{D}^{\mathrm{el}}\mathcal{A}^{-1} \tag{3}$$

where $D_{\vec{\alpha}_{\mathrm{el}}} = D + M^{\vec{\alpha}_{\mathrm{el}}}$ and $\vec{\alpha}_{\mathrm{el}} = -(\mathrm{i}(\omega + \varphi_{\mathrm{el}})e_1 + me_2)$. Thus the equation $\mathcal{D}^{\mathrm{el}}\Phi = 0$ is equivalent to the equation $D_{\vec{\alpha}_{\mathrm{el}}} F = 0$.

3.3. The Dirac equation with pseudoscalar potential

The Dirac equation with pseudoscalar potential has the form (see, e.g., [23])

$$\mathcal{D}^{\mathrm{ps}} = \mathcal{D} + \varphi_{\mathrm{ps}}\gamma_0\gamma_5$$

where φ_{ps} is a real-valued function. We have the following equality (see [17]), similar to (2) and (3):

$$D + \nu I + M^{\vec{\beta}} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{D}^{\mathrm{ps}}\mathcal{A}^{-1}$$

where $\nu = -\mathrm{i}\tilde{\varphi}_{\mathrm{ps}}$ and $\vec{\beta} = -(\mathrm{i}\omega e_1 + me_2)$.

Suppose that $\vec{\beta} \notin \mathfrak{S}$, that is $m^2 \neq \omega^2$ and denote $S^{\pm} = \frac{1}{2\lambda}M^{(\lambda \pm \vec{\beta})}$, where the complex number λ is chosen such that $\lambda^2 = \vec{\beta}^2$. We have [14]

$$D + \nu I + M^{\vec{\beta}} = S^+(D + (\nu + \lambda)I) + S^-(D + (\nu - \lambda)I). \tag{4}$$

The operators of multiplication S^+ and S^- are mutually complementary projection operators on the set of $\mathbb{H}(\mathbb{C})$-valued functions and commute with the operators in parentheses in (4). Thus f satisfies the equation

$$(D + \nu I + M^{\vec{\beta}})f = 0 \tag{5}$$

if and only if the functions $f^+ = S^+ f$ and $f^- = S^- f$ are solutions of the equations

$$(D + \nu + \lambda)f^+ = 0 \tag{6}$$

and

$$(D + \nu - \lambda)f^- = 0 \tag{7}$$

respectively. In other words, given f^+ and f^- solutions of (6) and (7), the function $f = S^+ f^+ + S^- f^-$ will be a solution of (5).

Equations (6) and (7) have a quite convenient form for studying, nevertheless in order to reduce them to the universal form (1) we make one additional step. We have

$$D + (\nu + \lambda)I = P_1^+(D + M^{(\nu+\lambda)\mathrm{i}e_1}) + P_1^-(D - M^{(\nu+\lambda)\mathrm{i}e_1}) \tag{8}$$

and

$$D + (\nu - \lambda)I = P_1^+(D + M^{(\nu-\lambda)\mathrm{i}e_1}) + P_1^-(D - M^{(\nu-\lambda)\mathrm{i}e_1}). \tag{9}$$

Thus f^+ and f^- are solutions of (6) and (7) respectively if and only if the functions $f^{++} = P_1^+ f^+$ and $f^{-+} = P_1^- f^+$ are solutions of the equations

$$(D + M^{(\nu+\lambda)\mathrm{i}e_1})f^{++} = 0 \tag{10}$$

$$(D - M^{(\nu+\lambda)\mathrm{i}e_1})f^{-+} = 0 \tag{11}$$

and the functions $f^{+-} = P_1^+ f^-$ and $f^{--} = P_1^- f^-$ are solutions of the equations

$$(D + M^{(\nu-\lambda)\mathrm{i}e_1})f^{+-} = 0 \qquad (D - M^{(\nu-\lambda)\mathrm{i}e_1})f^{--} = 0. \tag{12}$$

We sum up the obtained result in the following statement.

Proposition 1. *For $\vec{\beta} \notin \mathfrak{S}$ and $\lambda^2 = \vec{\beta}^2$ the following equality is valid*

$$D + \nu I + M^{\vec{\beta}} = P_1^+S^+(D + M^{(\nu+\lambda)\mathrm{i}e_1}) + P_1^-S^+(D - M^{(\nu+\lambda)\mathrm{i}e_1}) + P_1^+S^-(D + M^{(\nu-\lambda)\mathrm{i}e_1}) + P_1^-S^-(D - M^{(\nu-\lambda)\mathrm{i}e_1})$$

which implies

$$\ker(D + \nu I + M^{\vec{\beta}}) = P_1^+S^+\ker(D + M^{(\nu+\lambda)\mathrm{i}e_1}) \oplus P_1^-S^+\ker(D - M^{(\nu+\lambda)\mathrm{i}e_1}) \oplus P_1^+S^-\ker(D + M^{(\nu-\lambda)\mathrm{i}e_1}) \oplus P_1^-S^-\ker(D - M^{(\nu-\lambda)\mathrm{i}e_1})$$

where ker *means the set of null solutions in a domain of interest* $\Omega \subset \mathbb{R}^3$.

In this way equation (5) and hence the Dirac equation with pseudoscalar potential reduce to four equations of the form (1).

3.4. Force-free magnetic fields

Force-free magnetic fields appear as an important class of special solutions of non-linear equations of magnetohydrodynamics and are intensively studied in different branches of modern physics (see, e.g., [1, 6, 8, 10, 11, 20–22, 25, 26]). They are characterized by the following pair of equations:

$$\operatorname{div}\mathbf{B} = 0 \tag{13}$$

and

$$\operatorname{rot}\mathbf{B} + \nu\mathbf{B} = 0 \tag{14}$$

where ν is a scalar function. This system can be rewritten in the form

$$(D+\nu)\mathbf{B} = 0. \tag{15}$$

It will be more convenient for us to extend the class of solutions of (15) and to consider not only its purely vectorial solutions but more general complete biquaternionic functions. Thus we consider the equation

$$(D+\nu)f = 0 \tag{16}$$

where f is an $\mathbb{H}(\mathbb{C})$-valued function. Solutions of (13), (14) represent a subset of solutions of (16) which fulfil the additional requirement $\operatorname{Sc} f = 0$.

Now by analogy with the preceding subsection we obtain that f is a solution of (16) if and only if the functions $f^+ = P_1^+ f$ and $f^- = P_1^- f$ are solutions of the equations $(D + M^{\mathrm{i}\nu e_1})f^+ = 0$ and $(D - M^{\mathrm{i}\nu e_1})f^- = 0$, respectively.

Thus the system (13), (14) reduces to a pair of equations of the form (1).

3.5. Maxwell's equations for slowly changing media

Consider first the general Maxwell system

$$\operatorname{rot}\mathbf{H} = \varepsilon\partial_t\mathbf{E} + \mathbf{j} \tag{17}$$

$$\operatorname{rot}\mathbf{E} = -\mu\partial_t\mathbf{H} \tag{18}$$

$$\operatorname{div}(\varepsilon\mathbf{E}) = \rho \tag{19}$$

$$\operatorname{div}(\mu\mathbf{H}) = 0 \tag{20}$$

where ε and μ are assumed to be functions of spatial coordinates only. It can be rewritten in the following form [16, 17]:

$$(D + M^{\vec{\varepsilon}})\vec{E} = -\frac{1}{c}\partial_t\vec{H} - \frac{\rho}{\sqrt{\varepsilon}} \tag{21}$$

and

$$(D + M^{\vec{\mu}})\vec{H} = \frac{1}{c}\partial_t\vec{E} + \sqrt{\mu}\mathbf{j} \tag{22}$$

where $\vec{E} = \sqrt{\varepsilon}\mathbf{E}, \vec{H} = \sqrt{\mu}\mathbf{H}$, $c = 1/\sqrt{\varepsilon\mu}$, $\vec{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}$ and $\vec{\mu} = \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}}$.

In a sourceless time-harmonic situation we obtain the equations

$$D_{\vec{\varepsilon}}\vec{E} = \mathrm{i}\nu\vec{H} \qquad \text{and} \qquad D_{\vec{\mu}}\vec{H} = -\mathrm{i}\nu\vec{E}. \tag{23}$$

Here $\nu = \omega/c$.

The medium is said to be slowly changing when its properties change appreciably over distances much greater than the wavelength [2, 24]. Usually this is associated with the possibility of reducing the Maxwell equations (23) to the Helmholtz equations

$$(\Delta + \nu^2)\vec{E} = 0 \qquad \text{and} \qquad (\Delta + \nu^2)\vec{H} = 0.$$

It is easy to check that such a reduction is possible if only $|\vec{\varepsilon}|$ and $|\vec{\mu}|$ are considered as relatively very small and the terms containing the vectors $\vec{\varepsilon}$ and $\vec{\mu}$ are supposed to be negligible. Then (23) take the form

$$D\vec{E} = \mathrm{i}\nu\vec{H} \qquad \text{and} \qquad D\vec{H} = -\mathrm{i}\nu\vec{E}$$

and can be diagonalized. For the functions $\vec{\varphi} = \vec{E} + \mathrm{i}\vec{H}$ and $\vec{\psi} = \vec{E} - \mathrm{i}\vec{H}$ we obtain the equations

$$(D - \nu)\vec{\varphi} = 0 \qquad \text{and} \qquad (D + \nu)\vec{\psi} = 0.$$

Now, by analogy with equation (15) each of these two equations can be rewritten in the form (1).

3.6. The static Maxwell system

When the vectors of the electromagnetic field do not depend on time, from (21) and (22) we obtain two independent equations

$$(D + M^{\vec{\varepsilon}})\vec{E} = -\frac{\rho}{\sqrt{\varepsilon}} \tag{24}$$

and

$$(D + M^{\vec{\mu}})\vec{H} = \sqrt{\mu}\mathbf{j}. \tag{25}$$

In a sourceless situation both equations reduce to (1).

3.7. Two important cases

Let us resume the results presented in this section. We see that the six physical models considered here reduce to equation (1). In the case of the static Maxwell system $\vec{\alpha}$ is a gradient of some scalar function. In all other cases in general $\vec{\alpha}$ must not be necessarily a gradient, though such a possibility is not excluded. In the first five models only one of the components of $\vec{\alpha}$ is a function while the other two are constants. Thus we are interested basically in the following two situations.

1. $\vec{\alpha}$ is a gradient of some scalar function φ: $\vec{\alpha} = \nabla\varphi$.
2. $\vec{\alpha}$ has the form $\vec{\alpha} = \alpha_1(x_1, x_2, x_3)e_1 + \alpha_2 e_2 + \alpha_3 e_3$, where α_1 is a complex valued function and α_2, α_3 are complex constants (of course, when $\alpha_1 = \alpha_1(x_1)$ we have the first case again).

4. Factorization of the Schrödinger operator

4.1. Scalar Schrödinger equations

Consider the Schrödinger operator $-\Delta + v$ applied to a scalar function ϕ. Here v is some complex valued function. Let $\vec{\alpha}$ be a purely vectorial biquaternion valued function such that

$$D\vec{\alpha} + (\vec{\alpha})^2 = -v. \tag{26}$$

Then as was observed in [3] and [4], the following equality is valid:

$$(-\Delta + v)\phi = (D + M^{\vec{\alpha}})(D - M^{\vec{\alpha}})\phi. \tag{27}$$

This equality gives a certain relation between solutions of (1) and null-solutions of the Schrödinger operator. Moreover, a fundamental solution of the operator $D_{\vec{\alpha}}$ can be obtained if a fundamental solution of the Schrödinger operator is given. In some cases, for instance when $\vec{\alpha} = \alpha_1(x_1)e_1$, this leads to construction of integral representations for solutions of (1) (see [18]).

Equation (26) represents a generalization of the famous Riccati differential equation. In [12] (see also [17]) it was shown that it is not merely a formal resemblance. In contrast, the well-known Euler theorems, the Weyr theorem as well as the Picard theorem describing the unique properties of the Riccati equation were generalized for equation (26).

Equation (26) necessarily implies that $\vec{\alpha}$ is a gradient, because from the vector part of (26) we have that $\operatorname{rot}\vec{\alpha} = 0$. Thus it can be useful only for the first situation from subsection 3.7.

Let us consider the product of the operators $D_{\vec{\alpha}}$ and $D_{-\vec{\alpha}}$ in application to a twice differentiable biquaternion valued function u and for any differentiable purely vectorial biquaternion $\vec{\alpha}$. We have

$$\begin{aligned} D_{\vec{\alpha}} D_{-\vec{\alpha}} u &= (D + M^{\vec{\alpha}})(D - M^{\vec{\alpha}}) \sum_{k=0}^{3} u_k e_k \\ &= \sum_{k=0}^{3} \left(M^{e_k} \left(D + M^{\vec{\alpha}^{(k)}} \right) \left(D - M^{\vec{\alpha}^{(k)}} \right) u_k \right). \end{aligned}$$

For each component u_k we use (27) and note that $(\vec{\alpha}^{(k)})^2 = \vec{\alpha}^2$. We obtain

$$D_{\vec{\alpha}} D_{-\vec{\alpha}} u = \sum_{k=0}^{3} (-\Delta u_k - \vec{\alpha}^2 u_k - (D\vec{\alpha}^{(k)}) u_k) e_k. \tag{28}$$

From (28) we see that the equation $D_{\vec{\alpha}} D_{-\vec{\alpha}} u = 0$ is equivalent to four scalar Schrödinger equations if only $D\vec{\alpha}^{(k)}$ is scalar for any $k = 0, 1, 2, 3$. It is easy to verify that it is possible if only $\vec{\alpha}$ has the following form:

$$\vec{\alpha} = \alpha_1(x_1)e_1 + \alpha_2(x_2)e_2 + \alpha_3(x_3)e_3. \tag{29}$$

Let us consider this case in detail. First of all we note that for the Dirac operator with scalar, electric or pseudoscalar potential this restriction on $\vec{\alpha}$ implies that the potential is an arbitrary function of one spatial coordinate. In the case of force-free magnetic fields (subsection 3.4) the proportionality factor ν is a function of one variable. In the case of the electromagnetic field in a slowly changing medium (subsection 3.5) the wavenumber ν is a function of one variable. Such media are known as stratified media.

For the static Maxwell system $\operatorname{div}(\varepsilon \mathbf{E}) = 0$, and $\operatorname{rot} \mathbf{E} = 0$ we obtain that it is equivalent to (1) with $\vec{\alpha}$ having form (29) iff the permittivity ε has the following form $\varepsilon = \varepsilon_1(x_1)\varepsilon_2(x_2)\varepsilon_3(x_3)$, where $\varepsilon_1, \varepsilon_2$ and ε_3 are arbitrary functions. α_k are related to ε_k in the following way $\alpha_k = (\partial_k \varepsilon_k)/(2\varepsilon_k)$, $k = 1, 2, 3$.

Thus for all considered physical models $\vec{\alpha}$ in form (29) corresponds to quite interesting, non-trivial situations. Let us study in detail the structure of solutions of equation (1) when $\vec{\alpha}$ has the form (29).

Denote

$$v_k = -D\vec{\alpha}^{(k)} - \vec{\alpha}^2 \tag{30}$$

and

$$w_k = D\vec{\alpha}^{(k)} - \vec{\alpha}^2 \qquad k = 0, 1, 2, 3. \tag{31}$$

Proposition 2. *Let f be a solution of (1) with $\vec{\alpha}$ having the form (29). Then the components f_k are solutions of the Schrödinger equations*

$$(-\Delta + w_k) f_k = 0 \qquad k = 0, 1, 2, 3. \tag{32}$$

Proof. Assume that f is a solution of (1). Then considering the equation $D_{-\vec{\alpha}} D_{\vec{\alpha}} f = 0$ we arrive at equations (32) for the components f_k. □

The following statement gives us a method for constructing exact solutions of (1) having obtained solutions of the corresponding Schrödinger equations.

Proposition 3. *Let $\vec{\alpha}$ be of the form (29) and four scalar functions $g_k, k = 0, 1, 2, 3$ satisfy the following equations:*

$$(-\Delta + v_k) g_k = 0. \tag{33}$$

Then the function

$$f = (D - M^{\vec{\alpha}}) g \tag{34}$$

is a solution of (1), where $g = \sum_{k=0}^{3} g_k e_k$.

Proof. This is an immediate consequence of (28). □

Example 4. Let $\vec{\alpha} = \sum_{k=1}^{3} \frac{1}{x_k - b_k} e_k$, where b_k are arbitrary complex constants. We have

$$v_0 = -D\vec{\alpha} - \vec{\alpha}^2 = 0. \tag{35}$$

This means that taking any scalar harmonic function g_0 we are able to construct a solution of (1) in the form $f = (D - M^{\vec{\alpha}}) g_0$. Calculating v_1, v_2 and v_3 according to (30) we obtain

$$v_1 = 2\left(\frac{1}{(x_2 - b_2)^2} + \frac{1}{(x_3 - b_3)^2}\right) \tag{36}$$

$$v_2 = 2\left(\frac{1}{(x_1 - b_1)^2} + \frac{1}{(x_3 - b_3)^2}\right) \tag{37}$$

$$v_3 = 2\left(\frac{1}{(x_1 - b_1)^2} + \frac{1}{(x_2 - b_2)^2}\right). \tag{38}$$

We will not try to find general solutions of the Schrödinger equations with these potentials. We show instead how one can always obtain a class of exact solutions of (1) and of the corresponding Schrödinger equations (33) when $\vec{\alpha}$ has the form (29).

Let us look for a one-component solution of (1): $f = f_k e_k$. We have $D f_k e_k + f_k \vec{\alpha}^{(k)} e_k = 0$. That is

$$\frac{\nabla f_k}{f_k} = -\vec{\alpha}^{(k)}. \tag{39}$$

It is easy to see that for $\vec{\alpha}$ of the form (29) the functions f_k are

$$f_0 = \mathrm{e}^{-(\Lambda_1+\Lambda_2+\Lambda_3)} \qquad f_1 = \mathrm{e}^{-\Lambda_1+\Lambda_2+\Lambda_3} \qquad f_2 = \mathrm{e}^{\Lambda_1-\Lambda_2+\Lambda_3} \qquad f_3 = \mathrm{e}^{\Lambda_1+\Lambda_2-\Lambda_3}$$

where $\Lambda_k = \Lambda_k(x_k)$, $k = 1, 2, 3$ is an antiderivative of α_k. Thus the function $\sum_{k=0}^{3} c_k f_k e_k$ is a solution of (1), where c_k are arbitrary complex constants.

Concerning the Schrödinger equations (33) it is useful to consider the quaternionic Riccati equation (26). Any solution of it can be represented in the form $\vec{\alpha} = \frac{\nabla\varphi}{\varphi}$, where φ is a solution of the equation $-\Delta\varphi + v\varphi = 0$ (see [12, 17]). For $\vec{\alpha}^{(k)}$ from (39) we obtain that $\vec{\alpha}^{(k)} = \frac{\nabla\varphi_k}{\varphi_k}$, where $\varphi_k = 1/f_k$ and consequently the functions φ_k are solutions of (33) for a corresponding k.

Let us see what are the functions f_k and φ_k in example 4.

Example 5. For $\vec{\alpha}$ from the preceding example we have

$$f_0 = \frac{1}{(x_1-b_1)(x_2-b_2)(x_3-b_3)} \qquad f_1 = \frac{(x_2-b_2)(x_3-b_3)}{x_1-b_1}$$
$$f_2 = \frac{(x_1-b_1)(x_3-b_3)}{x_2-b_2} \qquad f_3 = \frac{(x_1-b_1)(x_2-b_2)}{x_3-b_3}.$$

Hence the function $f = \sum_{k=0}^{3} c_k f_k e_k$ is a solution of (1) and the functions

$$\varphi_0 = (x_1-b_1)(x_2-b_2)(x_3-b_3) \qquad \varphi_1 = \frac{x_1-b_1}{(x_2-b_2)(x_3-b_3)}$$
$$\varphi_2 = \frac{x_2-b_2}{(x_1-b_1)(x_3-b_3)} \qquad \varphi_3 = \frac{x_3-b_3}{(x_1-b_1)(x_2-b_2)}$$

are solutions of (33) with potentials (35)–(38), respectively.

Proposition 6. *Let $\vec{\alpha}$ be of the form (29) and $f = (D - M^{\vec{\alpha}})g$ a solution of (1), where $g = \sum_{k=0}^{3} g_k e_k$. Then necessarily g_k are solutions of (33), respectively.*

Proof. Consider

$$\begin{aligned}(D + M^{\vec{\alpha}})f &= (D + M^{\vec{\alpha}})(D - M^{\vec{\alpha}})\sum_{k=0}^{3} g_k e_k \\ &= \sum_{k=0}^{3}\left(M^{e_k}\left(D + M^{\vec{\alpha}^{(k)}}\right)\left(D - M^{\vec{\alpha}^{(k)}}\right)g_k\right) \\ &= \sum_{k=0}^{3}(M^{e_k}(-\Delta + v_k)g_k).\end{aligned}$$

From assumption of the proposition we obtain that $(-\Delta + v_k)g_k = 0$, $k = 0, 1, 2, 3$. □

Let Ω be a domain in $\mathbb{R}^3$ which in particular may coincide with the whole space. Let $F(\Omega)$ and $G(\Omega)$ be some functional spaces. The $\mathbb{H}(\mathbb{C})$-valued function f is said to belong to a functional space if each of its components f_k belongs to it.

Proposition 7. *Let $\vec{\alpha}$ be of the form (29). Assume that the equation $(-\Delta + w_k(\mathbf{x}))u(\mathbf{x}) = \mu(\mathbf{x})$, $\mathbf{x} \in \Omega$, $k = 0, 1, 2, 3$ is solvable for any $\mu \in F(\Omega)$ and the solution u belongs to $G(\Omega)$. Then the equation*

$$(D - M^{\vec{\alpha}})g = f \tag{40}$$

is solvable for any $f \in F(\Omega)$ and the solution g belongs to $\mathrm{im}D_{\vec{\alpha}}(G(\Omega))$.

Proof. Let u_k, $k = 0, 1, 2, 3$ be solutions of the equations $(-\Delta + w_k)u_k = f_k$. Then $g = (D + M^{\vec{\alpha}})\sum_{k=0}^{3} u_k e_k$ is a solution of (40). □

Remark 8. We do not specify the functional spaces here because the results on solvability of the inhomogeneous Schrödinger equation are numerous and correspond to very different situations. Let us give an example.

Example 9. Let $\Omega = \mathbb{R}^3$ and w has the form

$$w = \hat{w} - m^2 \tag{41}$$

where $\widehat{w} \in C_0^\infty(\mathbb{R}^3)$, $m > 0$. The equation $(-\Delta + w)u = \mu$ is uniquely solvable [7] for any $\mu \in L_{2,a}$ and $u \in H^2_{loc}(\mathbb{R}^3)$, $u = O\left(\frac{1}{|\mathbf{x}|}\right)$, $\frac{\partial u}{\partial |\mathbf{x}|} - \mathrm{i}mu = o\left(\frac{1}{|\mathbf{x}|}\right)$ for $|\mathbf{x}| \to \infty$. Here $L_{2,a}$ denotes the space of square integrable functions with support in a ball of radius a.

Thus from proposition 7 we have the solvability of (40) for any $\vec{\alpha}$ such that w_k defined by (31) have form (41), for example, for $\vec{\alpha} = \sum_{k=1}^3 \beta_k(x_k) + \mathrm{i}m_k$, where $\beta_k \in C_0^\infty(\mathbb{R}^3)$ and $m_k > 0$.

Proposition 10. *Under the conditions of proposition 7 any solution of (1) from $F(\Omega)$ has the form*

$$f = (D - M^{\vec{\alpha}})g \tag{42}$$

where $g = \sum_{k=0}^3 g_k e_k$ and g_k satisfy equations (33) in Ω.

Proof. From proposition 7 it follows that f can be represented in form (42), where $g \in \operatorname{im} D_{\vec{\alpha}}(G(\Omega))$. From proposition 6 we obtain that g_k are solutions of (33). □

For the Schrödinger operator there are developed different approaches for obtaining asymptotic fundamental solutions under some additional assumptions. A fundamental solution can be used for construction of a right-inverse operator for the Schrödinger operator which gives a possibility to solve inhomogeneous equations. In the following proposition we show that having constructed right-inverse operators for the Schrödinger operators $-\Delta + v_k$, $k = 0, 1, 2, 3$, one can construct a right-inverse operator for $D_{\vec{\alpha}}$.

Proposition 11. *Let $\vec{\alpha}$ be of the form (29) and T_k such operators that for any $\varphi \in F(\Omega)$: $(-\Delta + v_k)T_k\varphi = \varphi$ in Ω, $k = 0, 1, 2, 3$. Then for any $f = \sum_{k=0}^3 f_k e_k \in F(\Omega)$ we have $D_{\vec{\alpha}} T_{\vec{\alpha}} f = f$ in Ω, where $T_{\vec{\alpha}} f = (D - M^{\vec{\alpha}})\left(\sum_{k=0}^3 (T_k f_k)e_k\right)$.*

Proof. Consider

$$(D + M^{\vec{\alpha}})T_{\vec{\alpha}} f = \sum_{k=0}^3 \left(M^{e_k}\left(D + M^{\vec{\alpha}^{(k)}}\right)\left(D - M^{\vec{\alpha}^{(k)}}\right)(T_k f_k)\right)$$
$$= \sum_{k=0}^3 (M^{e_k}(-\Delta + v_k)T_k f_k) = \sum_{k=0}^3 f_k e_k = f.$$
□

4.2. Schrödinger equations with quaternionic potentials

Let us assume that $\vec{\alpha}$ has the form

$$\vec{\alpha} = \alpha_1(x_1, x_2, x_3)e_1 + \alpha_2 e_2 + \alpha_3 e_3 \tag{43}$$

where α_1 is an arbitrary complex valued differentiable function and α_2, α_3 are complex constants. In section 3 we saw that the first five physical models considered here correspond to this situation.

Note that in this case $D\vec{\alpha} = D\vec{\alpha}^{(1)} = -D\vec{\alpha}^{(2)} = -D\vec{\alpha}^{(3)}$. Hence from (28) we obtain

$$D_{\vec{\alpha}} D_{-\vec{\alpha}} u = -\Delta u - \vec{\alpha}^2 u - (D\vec{\alpha})(u_0 e_0 + u_1 e_1 - u_2 e_2 - u_3 e_3)$$

where $u = \sum_{k=0}^3 u_k e_k$.

Denote $Cu = -e_1ue_1 = u_0e_0+u_1e_1-u_2e_2-u_3e_3$, $Au = -\Delta u-\vec{\alpha}^2u$ and $Bu = -(D\vec{\alpha})u$. That is

$$D_{\vec{\alpha}}D_{-\vec{\alpha}} = A + BC \tag{44}$$

and

$$D_{-\vec{\alpha}}D_{\vec{\alpha}} = A - BC.$$

Denote $Q^{\pm} = \frac{1}{2}(I \pm \mathrm{i}e_1C)$.

Proposition 12. *Solutions of equation (1) (with $\vec{\alpha}$ of the form (43)) have the form $f = D_{-\vec{\alpha}}u$, where $u = Q^+v + Q^-w$ and v, w are solutions of the following Schrödinger equations with quaternionic potentials:*

$$(A + B\mathrm{i}e_1)v = 0 \tag{45}$$

and

$$(A - B\mathrm{i}e_1)w = 0 \tag{46}$$

respectively.

Proof. First, we note that $Q^{\pm}B = BQ^{\pm}$ and $Q^{\pm}C = \frac{1}{2}(C \pm \mathrm{i}e_1) = \frac{1}{2}(C\mathrm{i}e_1 \pm I)\mathrm{i}e_1 = \pm Q^{\pm}\mathrm{i}e_1$.

Applying Q^+ and Q^- to the equation

$$(A + BC)u = 0 \tag{47}$$

we see that it is equivalent to the pair of equations $(A + B\mathrm{i}e_1)Q^+u = 0$, $(A - B\mathrm{i}e_1)Q^-u = 0$, and Q^+, Q^- commute with the operators in parentheses and represent a pair of mutually complementary projection operators on the space of $\mathbb{H}(\mathbb{C})$-valued functions. Thus we have that u is a solution of (47) iff $u = Q^+v + Q^-w$ and v, w are solutions of (45) and (46) respectively. Now using (44) we finish the proof. □

Thus solution of (1) with $\vec{\alpha}$ of the form (43) reduces to solution of the Schrödinger equations with quaternionic potentials (45) and (46) which can be rewritten in a more explicit form as follows:

$$(-\Delta - (\vec{\alpha}^2 - \mathrm{i}D\alpha_1)I)v = 0 \tag{48}$$

and

$$(-\Delta - (\vec{\alpha}^2 + \mathrm{i}D\alpha_1)I)w = 0. \tag{49}$$

Remark 13. In the case when α_1 does not depend on x_1: $\alpha_1 = \alpha_1(x_2, x_3)$ it is easy to see that equations (48) and (49) are not independent. We have that if v is a solution of (48) then $w = \mathrm{i}e_1v$ is a solution of (49) and vice versa. Using this fact we obtain a one-to-one correspondence between solutions of (47) and solutions of (48). Such a correspondence is given by the operator

$$\Pi = \tfrac{1}{2}(I + \mathrm{i}e_1 - C + \mathrm{i}e_1C).$$

It can be verified immediately that $\Pi^2 = I$ and that u is a solution of (47) if and only if $v = \Pi u$ is a solution of (48) and vice versa. That is when $\vec{\alpha} = \alpha_1(x_2, x_3)e_1 + \alpha_2e_2 + \alpha_3e_3$ where α_2, α_3 are constants, solution of (1) reduces to solution of one Schrödinger equation with quaternionic potential (48).

Remark 14. Some classes of solutions of (48) and (49) can be obtained reducing the equations to scalar Schrödinger's equations using the following idea (proposed in [15] in another setting). Consider equation (48). If $(\mathrm{i}D\alpha_1 - \vec{\alpha}^2) \in \mathfrak{S}$ or $D\alpha_1 \in \mathfrak{S}$ then we can look for solutions of (48) in the form $v = (\mathrm{i}D\alpha_1 + \vec{\alpha}^2)f$ or $v = (D\alpha_1)f$ respectively where f is an unknown function. In the first case the equation reduces to the Laplace equation $\Delta v = 0$ and in the second to the Schrödinger equation $(\Delta + \vec{\alpha}^2)v = 0$.

Suppose that neither of these is the case. Then denote $\vec{\beta} = \mathrm{i}D\alpha_1$ and introduce β_0 as a scalar square root of $\vec{\beta}^2$. The complex quaternions $\beta = \beta_0 + \vec{\beta}$ and $\overline{\beta} = \beta_0 - \vec{\beta}$ are conjugate zero divisors. Equation (48) then can be rewritten as follows:

$$(-\Delta - (\beta_0 + \vec{\alpha}^2) + \beta)v = 0.$$

Looking for its solutions of the form $v = \bar{\beta}f$ we reduce it to the scalar Schrödinger equation $(\Delta + (\beta_0 + \vec{\alpha}^2))v = 0$.

5. Conclusions

We have shown that an ample class of physically meaningful partial differential systems of first order are equivalent to a single quaternionic equation which in its turn reduces in general to a Schrödinger equation with quaternionic potential, and in some situations considered in subsection 4.1 this last can be diagonalized. The rich variety of methods developed for different problems corresponding to the Schrödinger equation can be applied to the systems considered in the present work.

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